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Normal Modes for Massive Spin 1 Equation in Robertson–Walker Space-Time

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The general scheme of the massive spin-1 equation in curved space-time with associated conserved current is considered. A properly covariant scalar product between solutions of the spin-1 equation is defined in a standard way by means of the conservation of the current. The scheme is specialized to the Robertson–Walker space-time where solutions of the spin-1 equation have been previously determined. There results, in case of flat Robertson–Walker space-time, the automatic ortho-normalization property of the solutions, that therefore represent a set of normal modes.

KEY WORDS: massive spin 1 field; exact solutions; normal modes.

1. INTRODUCTION

The study of the spin 1 field equation in concrete examples of curved space-time is a subject of interest for different physical interpretations and applications. We recall that in the massless case the spin 1 field equation can be interpreted in terms of source free electromagnetic field (Penrose and Rindler, 1984) and in the massive case in terms of Proca fields (e.g. Illge, 1993). In view of a quantization of the theory, also the determination of the normal modes (for instance by the procedure induced by current conservation) is an argument of interest. In the massless case the spin 1 equation has been solved in the context of the Robertson–Walker space-time (Zecca, 1996a) and the corresponding electromagnetic interpretation given (Zecca, 1996b). The Proca fields interpretation has been discussed in Zecca (2005) by using the solution previously obtained in the massive case Zecca (2005).

In the present paper attention is devoted to the determination of the normal modes for spin 1 equation in Robertson–Walker space-time. To that end the enlarged scheme proposed by Illge (1993) in terms of four suitably related spinor fields is adopted as the staritng point. The solution of the equation previously

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determined is preliminary improved. In particular some further properties of the radial solutions and of the separated time equations are established.

A scalar product between solutions compatible with the principle of covariance in general relativity is then considered. It is defined in a general standard way by using a conserved current. In the flat space case the solutions of the spin 1 equation, that can be determined explicitly in terms of confluent hypergeometric functions, come out to be automatically orthogonal in the given scalar product. This fact is directly verified by using the properties of the conluent hypergeometric and spherical Bessel functions. It remains open the problem of the analogous calculation for the open and closed Robertson–Walker space-time where the analytical expression of the radial solutions has not yet been determined.

2. THE GENERAL EQUATION FOR THE SPIN 1 FIELD IN CURVED SPACE-TIME

It is usefull to first formulate the spin 1 equation in curved space-time following the scheme proposed by Illge (1993).

Accordingly the four spinor $\phi_{AB}, \chi_{A\dot{X}}, \theta_{A\dot{Y}}, \xi_{\dot{X}\dot{Y}}$ with $\phi_{AB} = \phi_{BA}, \xi_{\dot{X}\dot{Y}} = \xi_{\dot{Y}\dot{X}}$, are required to satisfy the equations

$$\nabla_{A\dot{X}}\phi^{A}_{B} = i\mu_{\star}\chi_{B\dot{X}}, \qquad \nabla_{A\dot{X}}\chi^{\dot{X}}_{B} = -i\mu_{\star}\phi_{AB} \tag{1}$$

$$\nabla_{A\dot{X}}\overline{\theta}^{\dot{X}}_{B} = -i\mu_{\star}\overline{\xi}_{AB}, \qquad \nabla_{A\dot{X}}\overline{\xi}^{A}_{B} = i\mu_{\star}\overline{\theta}_{\dot{X}B} \tag{2}$$

where $2\mu_{\star}^2 = m_0^2$, m_0 the mass of the particles of the field. The Eq. (2) have been directly written for $(\overline{\theta}, \overline{\xi})$ to point out that the system relative to (θ, ξ) is the complex conjugate of that for (ϕ, χ).

A general aspect of the scheme of Eqs. (1) and (2) is that the expression

$$J^{A\dot{X}}(\Phi, \Phi') = \phi^{A}_{B}\overline{\theta}^{\dot{X}B} + \chi^{\dot{X}}_{B}\overline{\xi}^{AB} - \left(\overline{\phi}^{\dot{X}}_{\dot{Y}}\theta^{A\dot{Y}} + \overline{\chi}^{A}_{\dot{Y}}\xi^{\dot{Y}\dot{X}}\right) \tag{3}$$

$(\Phi \equiv (\phi, \chi), \Phi' \equiv (\theta, \xi))$ can be interpreted as a current that is conserved

$$\nabla_{A\dot{X}}J^{A\dot{X}}(\Phi, \Phi') = 0 \tag{4}$$

This can be checked from the validity of Eqs. (1) and (2) and their complex conjugate. (One could also see that the term in parenthesis in Eq. (3) is separately conserved.)

3. SOLUTIONS OF THE SPIN 1 EQUATION IN ROBERTSON–WALKER SPACE-TIME

It is possible to solve the Eq. (1) in the Robertson–Walker space-time whose line element is

$$ds^2 = dt^2 - R(t)^2 \left[\frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2) \right], \quad a = 0, \pm 1. \tag{5}$$

Indeed the Eq. (1) can be solved by generalizing the separation method used to integrate the massless case (Zecca, 2005). The solution of Eq. (1) has therefore the form

$$\begin{aligned} &\phi_{00} = \alpha(t)\phi_0(r)S^{(0)}(\theta, \phi), && \chi_{00'} = A(t)\phi_1(r)S^{(1)}(\theta, \phi) = -\chi_{11'} \\ &\phi_{01} = \phi_{10} = \alpha(t)\phi_1(r)S^{(1)}(\theta, \phi), && \chi_{10'} = A(t)\phi_2(r)S^{(2)}(\theta, \phi) \\ &\phi_{11} = \alpha(t)\phi_2(r)S^{(2)}(\theta, \phi), && \chi_{01'} = -A(t)\phi_0(r)S^{(0)}(\theta, \phi), \end{aligned} \tag{6}$$

where the angular functions result to be $S^{(i)} \equiv S^{(i)}_{lm}(\theta, \varphi) = S_{ilm}(\theta)e^{im\varphi}(i = 0, 1, 2)$. The S_{ilm}'s $(l = 0, 1, 2, \ldots; m = 0, \pm 1, \pm 2, \ldots)$ have the properties $S_{1lm}(\theta) = S_{1l-m}(\theta)$, $S_{0lm}(\theta) = -S_{2l-m}(\theta)$, are essentially given by the Legendre functions or by the Jacobi polynomials and their expressions can be found in Zecca (1996a). Accordingly they can be assumed to be already ortho-normalized

$$\int d\Omega\, S^{(i)}_{lm}(\theta, \varphi) S^{(i)\star}_{l'm'}(\theta, \varphi) = \delta_{ll'}\delta_{mm'} \quad (i = 0, 1, 2) \tag{7}$$

The time dependence of the solutions of Eq. (1) is governed by the functions α, A, solutions of the coupled equations

$$ik\alpha = -\dot{\alpha}\, R - 2\alpha\dot{R} + im_0\, RA, \qquad ikA = \dot{A}\, R + A\dot{R} - im_0\, R\alpha. \tag{8}$$

By combining the Eq. (8) for different values of the separation constant one can derive the equation

$$i(k - k')(A_k\overline{\alpha}_{-k'} - \overline{A}_{-k'}\alpha_k) = \left(R\frac{d}{dt} + +3\dot{R} \right) (A_k\overline{\alpha}_{-k'} + \overline{A}_{-k'}\alpha_k). \tag{9}$$

In case $k = k'$, a simple integration gives

$$R^3\, (A_k\overline{\alpha}_{-k} + \overline{A}_{-k}\alpha_k) = \text{constant} = D \tag{10}$$

a property of the time solutions that will be usefull in the following. One has also

$$\alpha_k(t; -\mu_\star) = \overline{\alpha_{-k}(t; \mu_\star)}, \quad A_k(t; -\mu_\star) = \overline{A_{-k}(t; \mu_\star)} \tag{11}$$

that follow by taking into account the dependence on the mass of the particle into Eq. (8).

For what concerns he radial solutions ϕ_0, ϕ_2, they are such that $\phi_0 \cong \overline{\phi}_2$ while ϕ_0, ϕ_1 satisfy the equations

$$\begin{aligned} & r(1-ar^2)\phi_0'' + (4-5ar^2)\phi_0' \\ & \quad + \left[(k^2-3a)r + \frac{2-\lambda^2}{r} + 2ik(1-ar^2)^{\frac{1}{2}}\right]\phi_0 = 0 \qquad (12) \\ & r(1-ar^2)\phi_1'' + (4-5ar^2)\phi_1' + \left[(k^2-4a)r + \frac{2-\lambda^2}{r}\right]\phi_1 = 0. \end{aligned}$$

The parameters k and λ are the time and angular separation constants, respectively, and it comes out $\lambda^2 = l(l+1)$. The separated radial equations are difficult to be solved in general except for the flat space case $a = 0$ where solutions are explicitly given by

$$\phi_d(r) = r^{l-1} e^{ikr} M(l+2-d; 2l+2; -2ikr), \qquad d = 0, 1 \quad (a = 0) \qquad (13)$$

M the confluent hypergeometric function. The expression of the radial solutions of the flat case can be further simplified. From a special case of the confluent hypergeometric function (Abramovitz and Stegun, 1970) one has

$$\phi_{1k} \cong \frac{1}{r^{3/2}} J_{l+1/2}(-kr) \qquad (14)$$

$$\cong \frac{1}{r} j_l(-kr) \qquad (15)$$

the j_l's the spherical Bessel functions, solutions of the free particle radial Schrödinger equation in polar coordinates. As to ϕ_{0k} one has

$$\phi_{0k} \cong J_{l+3/2}(-kr) - \left(2i - \frac{1}{kr}\right) J_{l+1/2}(-kr) - J_{l-1/2}(-kr) \qquad (16)$$

$$\cong j_{l+1}(-kr) - \left(2i - \frac{1}{kr}\right) j_l(-kr) - j_{l-1}(-kr) \qquad (17)$$

$$\cong \left[-i\frac{d}{dz} - \frac{i}{z} + 1\right] j_l(z) \quad (z = -kr) \qquad (18)$$

To obtain Eq. (16) the second derivative and then a special case of the confluent hypergeometric function has been considered. The Eq. (18) follows from the recurrence relations of the spherical Bessel functions involving their derivatives (Abramovitz and Stegun, 1970).

4. THE NORMAL MODES

An argument of interest in view of a quantization of the spin 1 field in curved space-time is the determination of the normal modes (e.g. Birrell and

Davies, 1982). By the conserved current in hand this can be done by applying to the solution previously determined a standard procedure. From current conservation and Gauss' theorem (Hawking and Ellis, 1973) the expression

$$(\Phi, \Phi') = \int d_3x\, |g|^{1/2}\, J_\alpha(\Phi, \Phi') n^\alpha d\Sigma \tag{19}$$

$$= \int_{t=t_0} d_3x\, |g|^{1/2}\, \sigma^t_{AA'} J^{AA'}(\Phi, \Phi') \tag{20}$$

is independent of the Cauchy surface Σ. In the Robertson–Walker space-time Σ has been chosen to be the surface $t = t_0$ with normal unit vector $n^\alpha = (1, 0, 0, 0)$. The expression of the sigma matrix is $\sigma^t_{AA'} = \mathrm{diag}\{1/2, 1/2\}$. It can be obtained in a standard way (e.g. Chandrasekhar, 1993) from the null tetrad frame that was employed to obtain the solution of the Eq. (1) (Zecca, 1996a,b, 2005, 2006).

To calculate exlicitly the expression (15) on the solutions given in the previous section we note that $(\overline{\theta}, -\overline{\xi})$ satisfy the same equation satisfied by (ϕ, χ) with the substitution $\mu_\star \;\to\; -\mu_\star$. We choose therefore $\Phi' \equiv (\theta', \xi')$ so that

$$\overline{\theta}^{\dot{X}A}_{k'l'm'} \;\equiv\; \chi^{A\dot{X}}_{k'l'-m'} \qquad -\overline{\xi}^{AB}_{k'l'm'} \;\equiv\; \phi^{AB}_{k'l'-m'}. \tag{21}$$

where $\phi,\ \xi$ have the structure given in Eq. (6) with now $\alpha = \alpha_k(t; -\mu_\star),\ A = A_k(t; -\mu_\star)$. By taking into account the properties of the S_{ilm} functions and the Eqs. (11), (20) and (21) one finally gets

$$\begin{aligned}(\Phi_{klm}, \Phi_{k'l'm'}) &= -\frac{R^3}{\sqrt{2}}(A_k\overline{\alpha}_{-k'} + \overline{A}_{-k'}\alpha_k) \int_0^\infty \frac{r^2 dr}{\sqrt{1 - ar^2}} \int d\Omega \\ &\quad \times \Big[2\phi_{1kl}\overline{\phi}_{1k'l'} S^{(1)}_{lm} S^{(1)}_{l'm'} + \phi_{2kl}\overline{\phi}_{2k'l'} S^{(2)}_{lm} S^{(2)}_{l'm'} \\ &\quad + \phi_{0kl}\overline{\phi}_{0k'l'} S^{(0)}_{lm} S^{(0)}_{l'm'}\Big] + c.c. \end{aligned}\tag{22}$$

$$\begin{aligned}&= -\frac{R^3}{\sqrt{2}}\left(A_k\overline{\alpha}_{-k'} + \overline{A}_{-k'}\alpha_k\right) \int_0^\infty \frac{dr\ r^2}{\sqrt{1 - ar^2}} \left[2\phi_{1kl}\overline{\phi}_{1k'l'}\right. \\ &\quad \left. + \phi_{2kl}\overline{\phi}_{2k'l'} + \phi_{0kl}\overline{\phi}_{0k'l'}\right] \delta_{ll'}\delta_{mm'} + \ c.c. \end{aligned}\tag{23}$$

where also Eq. (7) has been taken into account. It seems difficult to calculate in general the integral in (23). However this is possible in the flat space case. Indeed, by taking into account the linearity of the radial equations, consider the functions $\sqrt{l(l+1)}\phi_1,\ k\phi_o,\ k\phi_2$ with $\phi_o = \overline{\phi}_2$, where $\phi_1,\ \phi_0$ are given by Eqs. (15) and (18). Then

$$\int_0^\infty drr^2 \left[2l(l+1)\phi_{1kl}\overline{\phi}_{1k'l} + kk'\phi_{0kl}\overline{\phi}_{0k'l} + kk'\phi_{2kl}\overline{\phi}_{2k'l}\right] \tag{24}$$

$$= 2kk' \int_0^\infty dr j_{lk} j_{lk'} r^2 - 2 \int_0^\infty dr\ j_{lk} \left[j''_{lk'} r^2 + 2r\ j'_{lk'} - l(l+1) j_{lk'} \right] \quad (25)$$

$$= 2kk' \left(1 + \frac{k'}{k}\right) \int_0^\infty j_l(-kr) j_l(-k'r) r^2 dr \quad (26)$$

$$= 2\pi\ \delta(k - k') \quad (27)$$

(It has been set $j_{lk} = j_l(-kr)$, $j'_{lk} = dj_l(-kr)/dr$). In Eq. (25) an integration by parts of a term $r\ (j_{lk} j_{lk'})'$ has been already performed exactly. Also an integration by parts of the term $2r^2(j'_{lk} j'_{lk'})$ has been done and its contour term neglected because $r^2(j_{lk} j'_{lk'}) \overset{r\to\infty}{\longrightarrow} \cos[kr + (l+1)\frac{\pi}{2}] \sin[k'r + (l+1)\frac{\pi}{2}]$ that vanishes in the sense of the distribution theory for $r \to \infty$ as it can be easily seen by reducing to sum of exponentials. The Eq. (26) follows by the fact that j_l is solution of the free particle radial Schrödinger equation (e.g. Merzbacher, 1970). To obtain Eq. (27), the closure relation of the spherical Bessel functions has been considered (Arfken and Weber, 1995).

Therefore, by properly choosing the constant D in Eq. (10), one has

$$(\Phi_{klm}, \Phi_{k'l'm'}) = \delta_{mm'} \delta_{ll'} \delta(k - k') \quad (28)$$

that ensures that the Φ's satisfy the defining properties of the Normal Modes.

As a conclusion, possible normal modes have been determined for the massive spin-1 equation in the Robertson–Walker space-time. It lacks the determination of the normal modes and their properties in the open and closed space-time. This seems a difficult task both by proceeding in the analytical determination of the radial solutions, both by trying to employ the radial equations themselves in the calculation of the integral in (23).

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REFERENCES

Abramovitz, W. and Stegun, I. E. (1970). *Handbook of Mathematical Functions*, Dover, New York.

Arfken, G. B. and Weber, H. J. (1995). *Mathematical Methods for Physicists*, Academic Press, San Diego.

Birrell, N. D. and Davies, P. C. W. (1982). *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge.

Chandrasekhar, S. (1983). *The Mathematical Theory of Black Holes*, Oxford University Press, New York.

Hawking, S. W. and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time*, Cambridge University Press, New York.

Illge, R. (1993). *Communications in Mathematical Physics* **158**, 433.
Merzbacher, E. (1970). *Quantum Mechanics*, Wiley, New York.
Penrose, R. and Rindler, W. (1984). *Spinors and Space-Time*, Cambridge University Press, Cambridge.
Zecca, A. (1996a). *International Journal of Theoretical Physics* **35**, 323.
Zecca, A. (1996b). *International Journal of Theoretical Physics* **35**, 2615.
Zecca, A. (2005). *Il Nuovo Cimento B* **120**, 513.
Zecca, A. (2006). *Proca Fields Interpretation of Spin 1 Equation in Robertson–Walker Space-Time*. General Relativity and Gravitation **38**, 837.